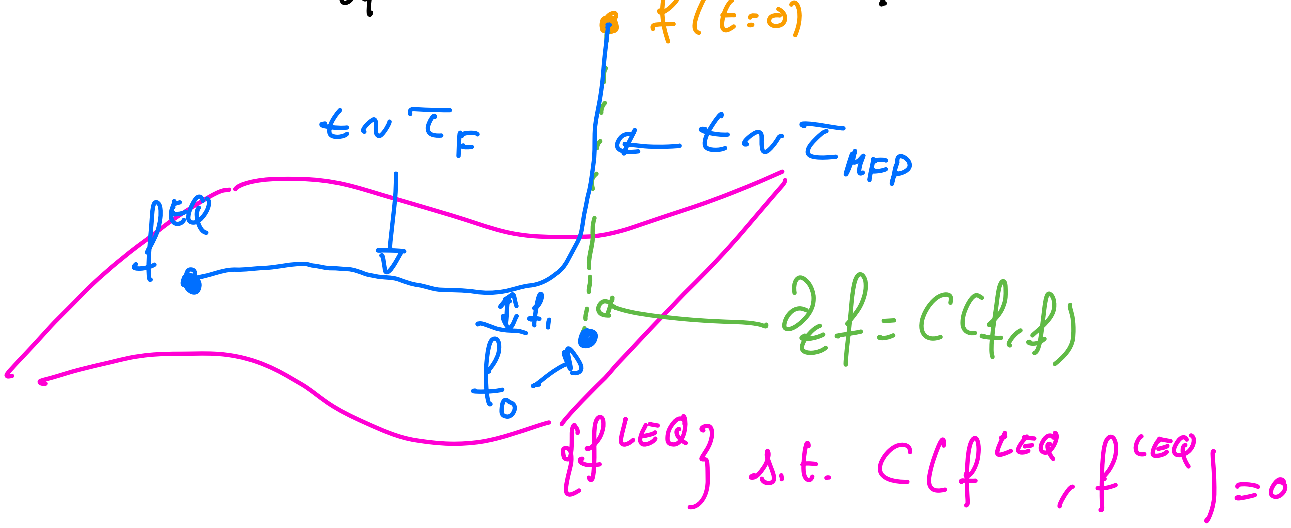
2.4.1) Relaxation of the phase space density

$$U=0 \qquad \partial_t f = -\vec{v}\cdot\frac{\partial f}{\partial \vec{q}} + C[f,f] \quad ; \quad \vec{v} = \frac{\vec{p}}{m}$$



$$\left.\begin{aligned} t &= \tau_F\,\hat{t} \\ C &= \frac{1}{\tau_{MFP}}\,\hat{C} \\ L_F f &= \tau_F\,\vec{v}\cdot\frac{\partial f}{\partial \vec{q}}\,f \\ \varepsilon &= \frac{\tau_{MFP}}{\tau_F} \end{aligned}\right\} \quad \partial_{\hat{t}} f = -L_F f + \frac{1}{\varepsilon}\hat{C}(f,f)$$

Perturbation theory $\quad f(\vec{q},\vec{p},t) = f_0(\vec{q},\vec{p},t) + \varepsilon\, f_1(\vec{q},\vec{p},t)$

Order by order

$$\mathcal{O}\left(\tfrac{1}{\varepsilon}\right): \quad C(f_0,f_0) = 0 \Rightarrow f_0 = f^{LEQ}(\vec{q},\vec{p},t)$$

$$\mathcal{O}(\varepsilon^0): \quad \partial_{\hat{t}} f_0 = -L_F f_0 + \hat{C}(f_1,f_0) + \hat{C}(f_0,f_1)$$

Leading order & hydrodynamic modes

$$f_0 \in \{f^{LEQ}\} \Rightarrow f_0 = \tilde{\sigma}(\vec{q},t)\, e^{-\vec{\alpha}(\vec{q},t)\cdot\vec{p} - \beta(\vec{q},t)\cdot\frac{\vec{p}^{\,2}}{2m}} \Rightarrow \text{Gaussian in } \vec{p}$$

Characterized by:

1. $\int d\vec{p}\, f(\vec{q},\vec{p},t) = n(\vec{q},t)$ average density field at $\vec{q}$ at time t

2. $\int d\vec{p}\, \vec{p}\, f(\vec{q},\vec{p},t) = w(\vec{q},t)$ average density of momentum field

3. $\frac{1}{2m}\int d\vec{p}\, \vec{p}^2 f(\vec{q},\vec{p},t) = K(\vec{q},t)$ average density of kinetic energy field

Indeed one conserved field entirely determine f_0!

To determine $f_0(\vec{q},\vec{p},t)$ we can use the time evolution of $n(\vec{q},t)$, $w(\vec{q},t)$ & $K(\vec{q},t)$.

2.4.2) The hydrodynamic equations

Evolution of the density field

$n(\vec{q},t) = \int d\vec{p}\, f(\vec{q},\vec{p},t) \Rightarrow \partial_t n$ from $\int d\vec{p}$ (BE)

$$\partial_t n = -\int d^3\vec{p}\; \vec{v}\cdot\frac{\partial}{\partial\vec{q}} f + \underbrace{\int d^3\vec{p}_1\, d^3\vec{p}_2\, d^2\sigma\, |\vec{v}_1-\vec{v}_2|\,(f_1'f_2'-f_1f_2)\times 1}_{\text{symmetrize by swapping } 1\leftrightarrow 2 \text{ \& then by swapping } \vec{p}_i \text{ \& } \vec{p}_i'}$$

$\vec{j}(\vec{q}) = \int d^3\vec{p}\, \vec{v} f$

$$= -\frac{\partial}{\partial\vec{q}}\cdot\vec{j}(\vec{q},t) + \frac{1}{4}\int d^3\vec{p}_1\, d^3\vec{p}_2\, d^2\sigma\, |\vec{v}_1-\vec{v}_2|\,(f_1'f_2'-f_1f_2)\underbrace{(\overbrace{1+1}^{1\leftrightarrow 2}\underbrace{-1-1}_{\vec{p}_i\leftrightarrow\vec{p}_i'})}_{=0}$$

$\partial_t n = -\vec{\nabla}\cdot\vec{j}$ local conservation law

Velocity field: $j \propto f$ which is extensive $\Rightarrow$ suggests defining a velocity field as $\vec{j} \equiv n(\vec{q},t)\times\vec{u}(\vec{q},t)$

③

$$\Leftrightarrow \quad \vec{u}(\vec{q},t) = \frac{\int d^3\vec{p}\; v\, f(\vec{q},\vec{p},t)}{\underbrace{\int d^3\vec{p}\; f(\vec{q},\vec{p},t)}_{\leftarrow\, n(\vec{q},t)}} = \int d^3\vec{p}\; v\; g(\vec{p}\,|\,\vec{q},t) \equiv \langle v \rangle_{f,\vec{q}} = \langle v \rangle$$

where $g(\vec{p}\,|\,\vec{q},t) = \dfrac{f(\vec{q},\vec{p},t)}{\int d^3\vec{p}\, f(\vec{q},\vec{p},t)}$ is the conditional probability density that a particle which is at $\vec{q}$ has a momentum $\vec{p}$.

Evolution equations: $\boxed{\partial_t n = -\partial_{\vec{q}} \cdot [n\, \vec{u}]}$

Material derivatives: $\Leftrightarrow \partial_t n + \vec{u}\cdot\partial_{\vec{q}}\, n = -n\, \partial_{\vec{q}}\cdot\vec{u}$

$\Leftrightarrow \boxed{D_t\, n = -n\, \partial_{\vec{q}}\cdot\vec{u}}$ with $\boxed{\begin{aligned} D_t &= \partial_t + \vec{u}\cdot\partial_{\vec{q}} \\ &= \partial_t + u_\alpha \partial_{q_\alpha}\end{aligned}}$

Evolution of the velocity field

$$u_\alpha = \frac{1}{n}\int dp\; v_\alpha f$$

$$\partial_t u_\alpha = -\frac{\partial_t n}{n^2}\underbrace{\int dv\; v_\alpha f}_{n\, u_\alpha} + \frac{1}{n}\int dp\, v_\alpha \overbrace{\partial_t f}^{[BE]}$$

$$= \frac{\partial_{q_\beta}(n u_\beta)}{n} u_\alpha + \frac{1}{n}\int d\vec{p}\; v_\alpha\left(-v_\beta \partial_{q_\beta} f\right) + \frac{1}{n}\int dp_1\, dp_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2|\,(f_1' f_2' - f_1 f_2)\; v_{1,\alpha}$$

symmetrization

$$\partial_t u_\alpha = u_\alpha \partial_{q_\beta} u_\beta + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta} n - \frac{1}{n}\partial_{q_\beta}\underbrace{\int d\vec{p}\; v_\alpha v_\beta f}_{n\langle v_\alpha v_\beta\rangle} + \frac{1}{n}\int dp_1\, dp_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2|\,(f_1' f_2' - f_1 f_2)\underbrace{\left[v_{1,\alpha} + v_{2,\alpha} - v'_{1,\alpha} - v'_{2,\alpha}\right]}_{=0 \text{ since } \vec{P}_{CM} = \vec{P}'_{CM}}$$

$$\partial_t u_\alpha + u_\beta \partial_{q_\beta} u_\alpha = (\dots) + u_\beta \partial_{q_\beta} u_\alpha$$

$$D_t u_\alpha = \partial_t u_\alpha + u_\beta \cdot \partial_{q_\beta} u_\alpha = \underbrace{\partial_{q_\beta}(u_\alpha u_\beta) + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta} n}_{\frac{1}{n}\partial_{q_\beta}(n u_\alpha u_\beta)} - \frac{1}{n}\partial_{q_\beta}\left(n \langle v_\alpha v_\beta \rangle\right)$$

$$D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta}\left[n\left(\langle v_\alpha v_\beta\rangle - \langle v_\alpha\rangle\langle v_\beta\rangle\right)\right] = -\frac{1}{n}\partial_{q_\beta}\left[n \langle v_\alpha v_\beta\rangle_c\right]$$

$$= -\frac{1}{n}\partial_{q_\beta}\left[n \langle \underbrace{(v_\alpha - \langle v_\alpha\rangle)}_{\equiv \delta v_\alpha = v_\alpha - u_\alpha}(v_\beta - \langle v_\beta\rangle)\rangle\right]$$

Introducing the pressure tensor $P_{\alpha\beta} = n\, m \langle \delta v_\alpha \delta v_\beta \rangle$ yields

$$\boxed{m D_t u_\alpha = -\frac{1}{n}\partial_{q_\alpha} P_{\alpha\beta}} \iff \boxed{m D_t \vec{u} = -\frac{1}{n}\vec{\nabla}\cdot\bar{\bar{P}}}$$

Comment: At this stage, we have the equivalent of Navier Stokes for a gas but f_0 is not fully determined $\Rightarrow$ need more information.

Kinetic energy density in the co-moving frame

$$\varepsilon(\vec{q}) = \langle \tfrac{1}{2} m\, \delta\vec{v}^{\,2}(\vec{q},t)\rangle = \frac{1}{2} m \left(\langle \vec{v}^{\,2}\rangle - \vec{u}^{\,2}\right)$$

Painful algebra leads to

$$\boxed{\partial_t \varepsilon + u_\alpha \partial_{q_\alpha}\varepsilon = -\frac{1}{n}\partial_\alpha h_\alpha - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta}}$$

where

* $\boxed{u_{\alpha\beta} = \frac{1}{2}\left(\partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha\right)}$ is the strain rate tensor
* $\boxed{h_\alpha = \frac{n\, m}{2}\langle \delta v_\alpha\, \delta v_\beta \delta v_\beta\rangle}$ is the kinetic energy flux

a.k.a. the heat flux $\vec{h} = \frac{n\, m}{2} \langle \delta\vec{v}\, \|\delta\vec{v}\|^2 \rangle$ 5

Temperature field

$T(\vec{q},t) = \frac{2}{3 k_B} \, \varepsilon(\vec{q},t)$ so that $\varepsilon = \frac{3}{2} k_B T$

$$\boxed{D_t T = \partial_t T + u_\alpha \partial_\alpha T = -\frac{2}{3 n k_B} \partial_\alpha h_\alpha - \frac{2}{3 n k_B} P_{\alpha\beta} u_{\alpha\beta}}$$

Closure:

f_0 is determined by $n, \vec{u}$ & T whose evolution is determined by $h_\alpha = \langle \frac{n\, m}{2} \delta\vec{v}\, \delta\vec{v}^2 \rangle$ & $P_{\alpha\beta} = n\, m \langle \delta v_\alpha \delta v_\beta \rangle$ that can be computed using $f_0 \Rightarrow$ dynamics is closed to leading order in ε.

2.4.3) leading order dynamics

Given $n, T, \vec{u}$, we determine f_0 through

$$\left.\begin{aligned} &\int f_0 \, d\vec{p} = n \\ &\int \vec{v} f_0 \, d\vec{p} = \vec{u} \\ &\int \frac{m}{2} (\vec{v} - \vec{u})^2 f_0 \, d\vec{p} = n \, \varepsilon(\vec{q},t) \\ &\qquad = \frac{3}{2} n k_B T(\vec{q},t) \end{aligned}\right\} \quad f_0 = \frac{n(\vec{q},t)}{(2\pi m k_B T(\vec{q},t))^{3/2}} e^{-\frac{[\vec{p} - m\vec{u}(\vec{q},t)]^2}{2 m k_B T(\vec{q},t)}}$$

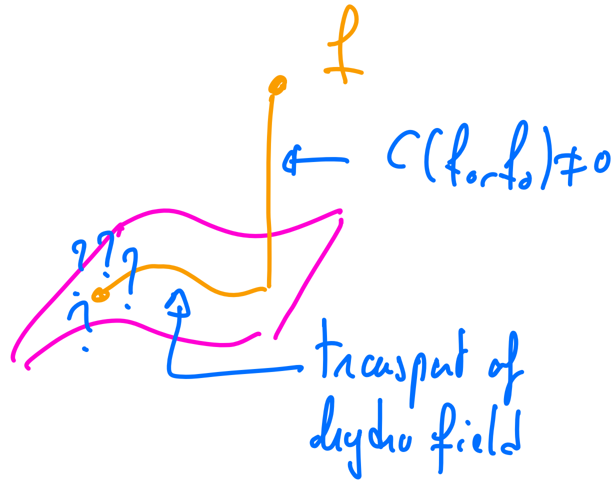
$$\boxed{f_0 = \frac{n}{(2\pi m k_B T)^{3/2}} e^{-\frac{m\, \delta\vec{v}^2}{2 k_B T}}}$$

Pressure & heat flux

$P^0_{\alpha\beta} = n_0 m \langle \delta v_\alpha \delta v_\beta \rangle_0 = n_0 m \frac{k_B T}{m} \delta_{\alpha\beta} = n_0 k_B T \delta_{\alpha\beta}$ Ideal gas law!

$$h^0_\alpha = \langle \delta v^2 \, \delta v_\alpha \rangle \frac{nm}{2} = 0 \quad \Rightarrow \text{odd moment!}$$

leading order hydrodynamics



$$\partial_t n_0 + \vec{u}_0 \cdot \vec{\nabla} n_0 = - n_0 \vec{\nabla} \cdot \vec{u}_0$$

$$m \left[\partial_t \vec{u}_0 + \vec{u}_0 \cdot \vec{\nabla} \vec{u}_0 \right] = - \frac{1}{n} \vec{\nabla} \cdot (n_0 k_B T)$$

$$\partial_t T_0 + \vec{u}_0 \cdot \vec{\nabla} T_0 = - \frac{2}{3} T_0 \vec{\nabla} \cdot \vec{u}_0$$

Why the "0"?

$$\begin{aligned} P &= n \, m \, \langle \delta v \otimes \delta v \rangle \\ &= m \int d\vec{p} \; \delta\vec{v} \otimes \delta\vec{v} \, f \quad ; \; f = f_0 + \varepsilon f_1 \\ &= \underbrace{m \int d\vec{p} \; \delta\vec{v} \otimes \delta\vec{v} \, f_0}_{\equiv P^0} + \underbrace{m \varepsilon \int d\vec{p} \; \delta\vec{v} \otimes \delta\vec{v} \, f_1}_{+ \varepsilon P_1} \end{aligned}$$